Journal of Mathematical Analysis and Applications **239**, 457–477 (1999)
Article ID jmaa.1999.6584, available online at http://www.idealibrary.com on IDEAL®

Hermite Polynomials in Asymptotic Representations of Generalized Bernoulli, Euler, Bessel, and Buchholz Polynomials

José L. López

Departamento de Matématica Aplicada, Facultad de Ciencias, Universidad de Zaragoza, 50013-Zaragoza, Spain

and

Nico M. Temme[1]

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Submitted by L. Debnath

Received April 8, 1999

This is the second paper on finite exact representations of certain polynomials in terms of Hermite polynomials. The representations have asymptotic properties and include new limits of the polynomials, again in terms of Hermite polynomials. This time we consider the generalized Bernoulli, Euler, Bessel, and Buchholz polynomials. The asymptotic approximations of these polynomials are valid for large values of a certain parameter. The representations and limits include information on the zero distribution of the polynomials. Graphs are given that indicate the accuracy of the first term approximations. © 1999 Academic Press

1. INTRODUCTION

Generalized Bernoulli, Euler, Bessel, and Buchholz polynomials of degree n, complex order μ and complex argument z, denoted respectively by $B_n^{\mu}(z)$, $E_n^{\mu}(z)$, $Y_n^{\mu}(z)$ and $P_n^{\mu}(z)$, can be defined by their generating

[1] E-mail: nicot@cwi.nl.

0022-247X/99 $30.00

functions [[11], chap. 6],

$$\frac{w^{\mu}e^{wz}}{(e^{w}-1)^{\mu}} = \sum_{n=0}^{\infty} \frac{B_n^{\mu}(z)}{n!} w^n, \qquad |w| < 2\pi, \tag{1}$$

$$\frac{2^{\mu}e^{wz}}{(e^{w}+1)^{\mu}} = \sum_{n=0}^{\infty} \frac{E_n^{\mu}(z)}{n!} w^n, \qquad |w| < \pi, \tag{2}$$

[[4], page 181],

$$\frac{1}{\sqrt{1-2zw}}\left(\frac{2}{1+\sqrt{1-2zw}}\right)^{\mu} e^{2w/(1+\sqrt{1-2zw})} = \sum_{n=0}^{\infty} \frac{Y_n^{\mu}(z)}{n!} w^n, \qquad |2zw| < 1, \tag{3}$$

and [[2], sec. 3], in a slightly different notation,

$$e^{z(\cot w - 1/w)/2}\left(\frac{\sin w}{w}\right)^{\mu} = \sum_{n=0}^{\infty} P_n^{\mu}(z) w^n, \qquad |w| < \pi. \tag{4}$$

The generalized Bernoulli and Euler polynomials play an important role in the calculus of finite differences. In fact, the coefficients in all the usual central-difference formulae for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of these polynomials [11]. Many properties of these polynomials can be found in [[3], chap. 6], [[5], vol. 1, chap. 1], [10], and [11]. An explicit formula for the generalized Bernoulli polynomials can be found in [12]. Asymptotic expansions in terms of elementary functions and in terms of gamma and polygamma functions are obtained in [16]. Properties and explicit formulas for the generalized Bernoulli and Euler numbers can be found in [9], [14], [15] and references therein.

The generalized Bessel polynomials form a set of orthogonal polynomials on the unit circle in the complex plane. They are important in certain problems of mathematical physics; for example, they arise in the study of electrical networks and when the wave equation is considered in spherical coordinates. For a historical survey and discussion of many interesting properties, we refer to [6]. New asymptotic expansions of $Y_n^{\mu}(x)$ (and its zeros) for large values of n are given in [17].

Buchholz polynomials are used for the representation of the Whittaker functions as convergent series expansions of Bessel functions [2]. They appear also in the convergent expansions of the Whittaker functions in ascending powers of their order and in the asymptotic expansions of the

Whittaker functions in descending powers of their order [7]. Explicit formulas for obtaining these polynomials may be found in [1].

In our first paper [8] it has been shown that Jacobi, Gegenbauer, Laguerre, and Tricomi-Carlitz polynomials have asymptotic representations in terms of the Hermite polynomials

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(2x)^{n-2k}.$$

These asymptotic representations include well-known limits of the polynomials in terms of the Hermite polynomials, and provide a powerful tool for approximating the zeros of these polynomials in terms of the zeros of the Hermite polynomials in the asymptotic limit [8].

The polynomials of the previous paper are all orthogonal on a set of the real line. The present group is quite different. Only the Bessel polynomials are orthogonal, but not in the standard sense: they are orthogonal on the unit circle. In a certain sense the polynomials of the present group become orthogonal if the parameter μ becomes large. We give similar asymptotic representations for the new group as in the previous paper for the asymptotic limit $|\mu| \to \infty$. From these representations we can derive

$$\lim_{\mu\to\infty} \left(\frac{24}{\mu}\right)^{n/2} B_n^{\mu}\left(\frac{\mu}{2} + \sqrt{\frac{\mu}{6}}\,z\right) = H_n(z),$$

$$\lim_{\mu\to\infty} \left(\frac{8}{\mu}\right)^{n/2} E_n^{\mu}\left(\frac{\mu}{2} + \sqrt{\frac{\mu}{2}}\,z\right) = H_n(z),$$

$$\lim_{\mu\to\infty} i^n(2\mu)^{n/2} Y_n^{\mu}\left[-\frac{2}{\mu}\left(1 + i\sqrt{\frac{2}{\mu}}\,z\right)\right] = H_n(z),$$

$$\lim_{\mu\to\infty} \left(\frac{6}{\mu}\right)^{n/2} P_n^{\mu}\left(-2\sqrt{6\mu}\,z\right) = \frac{1}{n!}H_n(z).$$

From these limits we can obtain approximations for the zeros of these polynomials in the asymptotic regime.

In the following section we give the principles of the Hermite-type asymptotic approximations used in this paper. In later sections we apply the method to obtain expansions for the generalized Bernoulli and Euler polynomials, and the Bessel and Buchholz polynomials. We also obtain estimates of their zeros for large μ.

2. EXPANSIONS IN TERMS OF HERMITE POLYNOMIALS

The Hermite polynomials have the generating functi

$$e^{2zw-w^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} w^n, \qquad z, w \in$$

which gives the Cauchy-type integral

$$H_n(z) = \frac{n!}{2\pi i} \int_{\mathscr{C}} e^{2zw-w^2} \frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin and the integr direction.

The polynomials defined in (1)–(4), as well as man polynomials, may be defined by a generating function

$$F(z,w) = \sum_{n=0}^{\infty} p_n(z) w^n,$$

where F: $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is analytic with respect to w contains the origin. We assume that $F(z,0) = p_0(z$ polynomials $p_n(z)$ are independent of w.

We have the Cauchy-type integral representation

$$p_n(z) = \frac{1}{2\pi i} \int_{\mathscr{C}} F(z,w) \frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin inside the analytic (as a function of w).

We write the generating function $F(z,w)$ of $p_n(z)$ i

$$F(z,w) = e^{A(z)w - B(z)w^2} f(z,w),$$

where $A(z)$ and $B(z)$ are independent of w, and it fc

$$p_n(z) = \frac{1}{2\pi i} \int_{\mathscr{C}} e^{A(z)w - B(z)w^2} f(z,w) \frac{d}{w}$$

The function f is also analytic around the origin $w =$ expand

$$f(z,w) = 1 + [p_1(z) - A(z)]w$$
$$+ \left[p_2(z) - A(z)p_1(z) + B(z) + \frac{1}{2} \right.$$
$$= \sum_{k=0}^{\infty} c_k(z) w^k.$$

Substituting this in (8) and using (5), we obtain the finite expansion

$$p_n(z) = (B(z))^{n/2} \sum_{k=0}^{n} \frac{c_k(z)}{(B(z))^{k/2}} \frac{H_{n-k}(\zeta)}{(n-k)!}, \qquad \zeta = \frac{A(z)}{2\sqrt{B(z)}}, \tag{10}$$

because terms with $k > n$ do not contribute to the integral in (8). If $B(z)$ happens to be zero for a special z-value, say z_0, we write

$$p_n(z_0) = [A(z_0)]^n \sum_{k=0}^{n} \frac{c_k(z_0)}{[A(z_0)]^k (n-k)!}. \tag{11}$$

In the examples considered in the following sections, the choice of $A(z)$ and $B(z)$ is based on our requirement that $c_1(z) = c_2(z) = 0$, in order to make the function $f(z,w)$ close to 1 near the origin [note that $f(z,0) = 1$]. Then, the generating function $F(z,w)$ is close to the generating function of the Hermite polynomials. Using $c_0(z) = 1$ and requiring $c_1(z) = c_2(z) = 0$, we have, from (9),

$$A(z) = p_1(z), \qquad B(z) = \tfrac{1}{2}p_1^2(z) - p_2(z). \tag{12}$$

We can summarize the above discussion in the following.

PROPOSITION 2.1. *Consider the polynomials $p_n(z)$ defined in (6) by a generating function $F(z,w)$ analytic in $w = 0$ and normalized in the form $F(z,0) = 1$. Then, they may be represented as the finite sum (10) if $p_1^2(z) - 2p_2(z) \neq 0$, and as the finite sum (11) if $p_1^2(z_0) - 2p_2(z_0) = 0$. The functions $c_k(z)$ are the coefficients of the Taylor expansion of*

$$F(z,w)\exp\left\{\left(\tfrac{1}{2}p_1(z)^2 - p_2(z)\right)w^2 - p_1(z)w\right\}$$

at $w = 0$, $c_0 = 1$, $c_1 = c_2 = 0$ and H_n are the Hermite polynomials.

In the following sections we verify if the finite sum in (10) yields asymptotic representations for the generalized Bernoulli, Euler, Bessel, and Buchholz polynomials. The special choice of $A(z)$ and $B(z)$ given in (12) is crucial for obtaining asymptotic properties. To prove these properties we will use the following lemma.

LEMMA 2.1. *Let $\phi(w)$ be analytic at $w = 0$, with Maclaurin expansion of the form $\phi(w) = \mu w^n(a_0 + a_1 w + a_2 w^2 + \cdots) + b_1 w + b_2 w^2 + \cdots$, where n is a positive integer, a_k, b_k are complex numbers that do not depend on the complex number μ, $a_0 \neq 0$; let c_k denote the coefficients of the power series of $e^{\phi(w)}$, that is, $e^{\phi(w)} = \sum_{k=0}^{\infty} c_k w^k$. Then*

$$c_k = \mathcal{O}\left(|\mu|^{\lfloor k/n \rfloor}\right), \qquad \mu \to \infty.$$

Proof. The proof follows from expanding $e^{\phi(w)} = \Sigma$ substituting the power series of ϕ and collecting equal pow

3. GENERALIZED BERNOULLI POLYNOM

From (1) we obtain the following Cauchy-type integral f ized Bernoulli polynomials

$$B_n^{\mu}(z) = \frac{n!}{2\pi i}\int_{\mathscr{C}} \frac{w^{\mu}e^{wz}}{(e^w - 1)^{\mu}}\frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin with radius less than that $F(z,w) = w^{\mu}e^{wz}/(e^w - 1)^{\mu}$ assumes real values for r w and μ.

We have

$$B_0^{\mu}(z) = 1, \qquad B_1^{\mu}(z) = z - \frac{\mu}{2}, \qquad B_2^{\mu}(z) = z^2 - \mu z -$$

Hence, by (12) and $p_n(z) = B_n^{\mu}(z)/n!$,

$$A(z) = z - \frac{\mu}{2}, \qquad B(z) = \frac{\mu}{24},$$

and by (10),

$$B_n^{\mu}(z) = n!\left(\frac{\mu}{24}\right)^{n/2} \sum_{k=0}^{n} \frac{c_k(\mu)}{(n-k)!}\left(\frac{24}{\mu}\right)^{k/2} H_{n-k}\left(\frac{\sqrt{6}(}{}\right.$$

Observe that this representation shows the symmetry respect to the point $z = \frac{1}{2}\mu$.

The coefficients $c_k(\mu)$ of the expansion are given i lemma.

LEMMA 3.1. *The odd coefficients $c_n(\mu)$ in the expansion*

$$c_{2n+1}(\mu) = 0 \qquad \forall n \geq 0,$$

and the even ones are independent of z; they are given for $n \geq 2$ by the recurrence

$$c_{2n}(\mu) = \frac{\mu}{12n} \sum_{k=2}^{n} \frac{(2k+1)(k-3)+6}{(2k+1)!} c_{2(n-k)}(\mu) - \frac{1}{n} \sum_{k=1}^{n} \frac{(n-k)}{(2k+1)!} c_{2(n-k)}(\mu), \tag{15}$$

with $c_0(\mu) = 1$, $c_2(\mu) = 0$ and satisfy

$$c_{2n}(\mu) = \mathcal{O}(\mu^{\lfloor n/2 \rfloor}), \qquad |\mu| \to \infty. \tag{16}$$

Proof. Using equation (7), the function $f(z,w)$ of the generalized Bernoulli polynomials reads

$$f(z,w) = \frac{w^{\mu} e^{\mu(1+w/12)w/2}}{(e^{w}-1)^{\mu}} = \sum_{k=0}^{\infty} c_k(\mu) w^k. \tag{17}$$

This is an even function in the variable w and (14) follows. It is independent of z and so are the coefficients c_k (which only depend on μ). Moreover, it satisfies the differential equation

$$\mu\left[1 + \left(\frac{1}{2} - \frac{1}{w} - \frac{w}{12}\right)(e^{w}-1)\right] f(z,w) + (e^{w}-1)\frac{d}{dw} f(z,w) = 0,$$

and introducing the expansion (17) [with $c_{2n+1}(\mu) = 0$] in this differential equation we obtain (15). The function $f(z,w)$ can be written in this case in the form

$$f(z,w) = e^{\mu w^4 \phi_1(w)}, \qquad \phi_1(w) = \frac{1}{2880} + \mathcal{O}(w^2), \qquad w \to 0,$$

and $\phi_1(w)$ does not depend on μ and z. Hence, the proof of (16) follows from Lemma 2.1. ∎

PROPOSITION 3.1. *The generalized Bernoulli polynomials $B_n^{\mu}(z)$ have the finite expansion in terms of Hermite polynomials*

$$B_n^{\mu}(z) = \left(\frac{\mu}{24}\right)^{n/2} H_n(\zeta) + n!\left(\frac{\mu}{24}\right)^{n/2} \sum_{k=2}^{\lfloor n/2 \rfloor} c_{2k}(\mu) \left(\frac{24}{\mu}\right)^{k} \frac{H_{n-2k}(\zeta)}{(n-2k)!}, \tag{18}$$

where

$$\zeta = \frac{\sqrt{6}\,(z - \mu/2)}{\sqrt{\mu}} \tag{19}$$

and $c_{2k}(\mu)$ *are given in* (15). *This is actually an asymptotic expansion of* $B_n^{\mu}(z)$ *for* $|\mu| \to \infty$ *with respect to the sequence* $\mu^{\lfloor k/2 \rfloor - k}$, *uniformly with respect to* ζ.

Proof. (18) follows trivially by using (13) and Lemma 3.1. The asymptotic property of (18) follows from (16). If $|\zeta|$ is bounded, the combination $c_{2k}(\mu)\mu^{-k}$ in (18) gives the asymptotic nature for large values of $|\mu|$; if $|\zeta|$ is not bounded, then the property $H_n(\zeta) = \mathcal{O}(\zeta^n)$ gives extra asymptotic convergence in the sum in (18). ∎

Figure 1 shows the accuracy of the approximation

$$B_n^{\mu}(z) \simeq \left(\frac{\mu}{24}\right)^{n/2} H_n\left(\frac{\sqrt{6}\,(z - \mu/2)}{\sqrt{\mu}}\right) \tag{20}$$

for $n = 10$, real z and several values of μ.

3.1. *Approximating the Zeros*

When computing approximations of the zeros of the generalized Bernoulli polynomials for large values of μ we start with the zeros of the Hermite polynomial $H_n(\zeta)$ in (20).

Let $b_{n,m}$ and $h_{n,m}$ be the mth zero of $B_n^{\mu}(z)$ and $H_n(z)$, respectively, $m = 1, 2, \ldots, n$. Then, for given μ and n we take the relation for ζ given in (19) to compute a first approximation of $b_{n,m}$ by writing

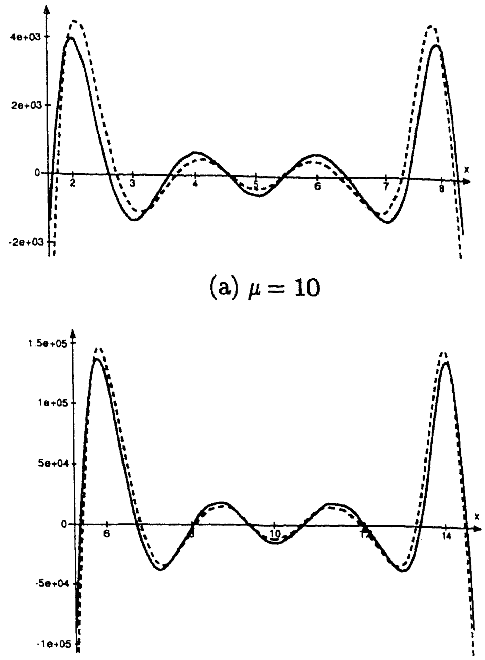
$$b_{n,m} \sim \frac{\mu}{2} + \sqrt{\frac{\mu}{6}}\, h_{n,m}.$$

For $\mu = 10, 20, 30$ and $n = 10$, the best relative accuracy in the zeros is $\sim 10^{-3}$ and the worst result (for the largest zero) is $\sim 10^{-2}$. For $\mu = 40, 50$ it oscillates between 10^{-3} and 10^{-4}, whereas for $\mu = 100$ it oscillates between 10^{-4} and 10^{-5}.

4. GENERALIZED EULER POLYNOMIALS

From (2) we obtain the Cauchy-type integral for the generalized Euler polynomials

$$E_n^{\mu}(z) = \frac{n!}{2\pi i} \int_{\mathscr{C}'} \frac{2^{\mu} e^{wz}}{(e^w + 1)^{\mu}} \frac{dw}{w^{n+1}},$$



(a) $\mu = 10$

(b) $\mu = 20$

FIG. 1. Solid lines represent $B_{[0}^{\mu}(x)$ for several values of μ, whereas dashed lines represent the right-hand side of (20).

where $\mathscr{C}$ is a circle around the origin with radius less than π. We assume that $F(z,w) = 2^{\mu}e^{wz}/(e^{w} + 1)^{\mu}$ assumes real values for real values of z, w, and μ.

We have

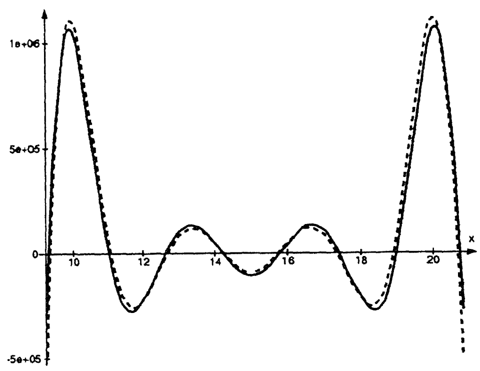
$$E_0^{\mu}(z) = 1, \qquad E_1^{\mu}(z) = z - \frac{\mu}{2}, \qquad E_2^{\mu}(z) = z^2 - \mu z + \frac{\mu(\mu - 1)}{4}.$$

Hence, by (12) and $p_n(z) = E_n^{\mu}(z)/n!$,

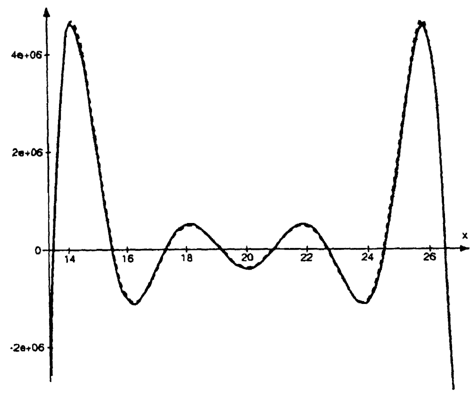
$$A(z) = z - \frac{\mu}{2}, \qquad B(z) = \frac{\mu}{8}.$$

It follows from (10) that

$$E_n^{\mu}(z) = n!\left(\frac{\mu}{8}\right)^{n/2} \sum_{k=0}^{n} \frac{c_k(\mu)}{(n-k)!}\left(\frac{8}{\mu}\right)^{k/2} H_{n-k}\left(\frac{\sqrt{2}(z - \mu/2)}{\sqrt{\mu}}\right), \tag{21}$$



(c) $\mu = 30$



(d) $\mu = 40$

FIG. 1. Continued.

where the coefficients $c_k(\mu)$ of the expansion are given below. This representation shows the symmetry of $E_n^{\mu}(z)$ with respect to the point $z = \frac{1}{2}\mu$.

We have the following results. The proofs are as in the case of the Bernoulli polynomials.

LEMMA 4.1. *The odd coefficients $c_n(\mu)$ in the expansion* (21) *vanish,*

$$c_{2n+1}(\mu) = 0 \qquad \forall n \geq 0, \tag{22}$$

and the even ones are independent of z; they are given for $n \geq 2$ by the recurrence

$$c_{2n}(\mu) = \frac{\mu}{16n} \sum_{k=2}^{n} \frac{2k-3}{(2k-1)!} c_{2(n-k)}(\mu) + \frac{1}{2n} \sum_{k=2}^{n} \frac{(k-n-1)}{(2k-2)!} c_{2(n-k+1)}(\mu), \tag{23}$$

where $c_0(\mu) = 1$, $c_2(\mu) = 0$, and satisfy

$$c_{2n}(\mu) = \mathcal{O}(\mu^{\lfloor n/2 \rfloor}), \qquad |\mu| \to \infty. \tag{24}$$

PROPOSITION 4.1. *The generalized Euler polynomials $E_n^{\mu}(z)$ have the finite expansion in terms of Hermite polynomials*

$$E_n^{\mu}(z) = \left(\frac{\mu}{8}\right)^{n/2} H_n(\zeta) + n!\left(\frac{\mu}{8}\right)^{n/2} \sum_{k=2}^{\lfloor n/2 \rfloor} c_{2k}(\mu)\left(\frac{8}{\mu}\right)^{k} \frac{H_{n-2k}(\zeta)}{(n-2k)!}, \tag{25}$$

where

$$\zeta = \frac{\sqrt{2}\,(z - \mu/2)}{\sqrt{\mu}} \tag{26}$$

and $c_k(\mu)$ are given in (23). *This is actually an asymptotic expansion of $E_n^{\mu}(z)$ for $|\mu| \to \infty$ with respect to the sequence $\mu^{\lfloor k/2 \rfloor - k}$, uniformly with respect to ζ.*

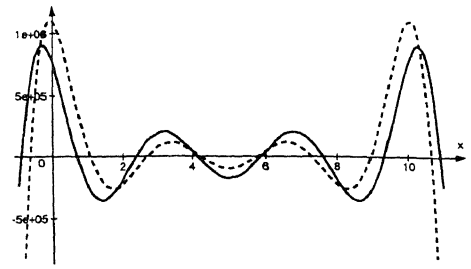
Figure 2 shows the accuracy of the approximation

$$E_n^{\mu}(z) \simeq \left(\frac{\mu}{8}\right)^{n/2} H_n\left(\frac{\sqrt{2}\,(z - \mu/2)}{\sqrt{\mu}}\right) \tag{27}$$

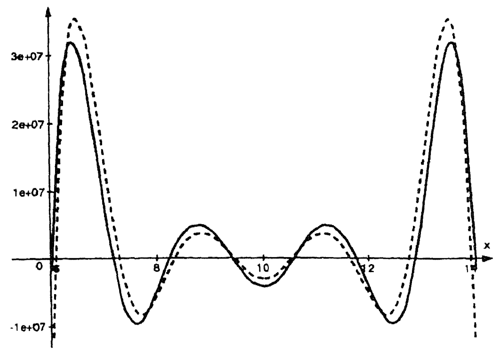
for $n = 10$, real z and several values of μ.

4.1. *Approximating the Zeros*

Let $e_{n,m}$ and $h_{n,m}$ be the mth zero of $E_n^{\mu}(z)$ and $H_n(z)$, respectively, $m = 1, 2, \ldots, n$. Then, for given μ and n we take the relation for ζ given



(a) $\mu = 10$



(b) $\mu = 20$

FIG. 2. Solid lines represent $E_{10}^{\mu}(x)$ for several values of μ, whereas dashed lines represent the right-hand side of (27).

in (26) to compute a first approximation of $e_{n,m}$ by writing

$$e_{n,m} \sim \frac{\mu}{2} + \sqrt{\frac{\mu}{2}}\, h_{n,m}.$$

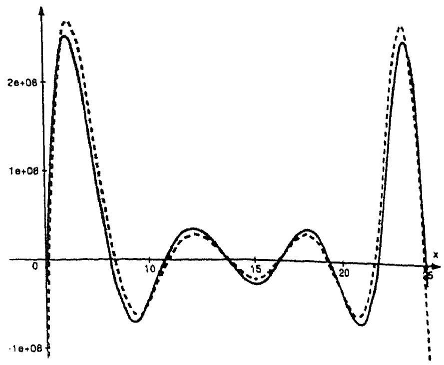
The accuracy is as in the case of the generalized Bernoulli polynomials.

5. GENERALIZED BESSEL POLYNOMIALS

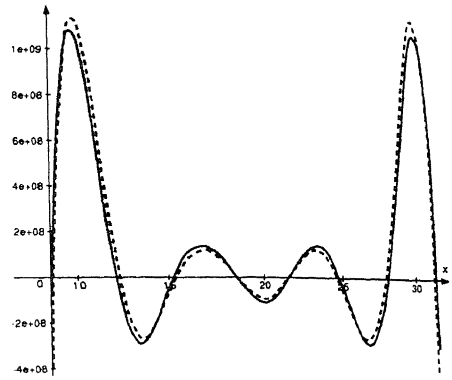
From (3) we obtain the following Cauchy-type integral for the generalized Bessel polynomials

$$Y_n^{\mu}(z) = \frac{n!}{2\pi i}\int_{\mathscr{C}} \frac{1}{\sqrt{1-2zw}}\left(\frac{2}{1+\sqrt{1-2zw}}\right)^{\mu} e^{2w/(1+\sqrt{1-2zw})}\frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin with radius less than $|1/(2z)|$.



(c) $\mu = 30$



(d) $\mu = 40$

FIG. 2. Continued.

We have

$$Y_0^{\mu}(z) = 1, \qquad Y_1^{\mu}(z) = \tfrac{1}{2}[2 + (\mu + 2)z],$$
$$Y_2^{\mu}(z) = \tfrac{1}{4}\left[4 + 4(\mu + 3)z + \{\mu(\mu + 7) + 12\}z^2\right].$$

Hence, by (12) and $p_n(z) = Y_n^{\mu}(z)/n!$,

$$A(z) = \frac{1}{2}\left[2 + (\mu + 2)z\right], \qquad B(z) = -\frac{z}{8}\left[4 + (3\mu + 8)z\right].$$

It follows from (10) that

$$Y_n^\mu(z) = n![B(z)]^{n/2} \sum_{k=0}^{n} \frac{c_k(z,\mu)}{[B(z)]^{k/2}} \frac{H_{n-k}(\zeta)}{(n-k)!}, \qquad \zeta = \frac{A(z)}{2\sqrt{B(z)}} \tag{28}$$

where the coefficients $c_k(z, \mu)$ of the expansion satisfy the properties given in the following lemma. We introduce the notation

$$y(w) = \sqrt{1-2zw} = 1 - \sum_{k=1}^{\infty} z^k b_k w^k, \qquad b_k = \frac{2^k}{2k!}\left(\frac{1}{2}\right)_{k-1}.$$

LEMMA 5.1. *The coefficients $c_k(z, \mu)$ in the expansion* (28) *are given by the recursion relation*

$$\begin{aligned} 16(k+1)c_{k+1} &= 48kzc_k - 28(k-1)z^2c_{k-1} + 4[6 + (18 - k + 5\mu)z]z^2c_{k-2} \\ &\quad - [32 + (64 + 3(k-3) + 25\mu)z]z^3c_{k-3} \\ &\quad + 2\sum_{j=0}^{k-4}\Big[(4jb_{k+1-j} \\ &\qquad - 8jb_{k-j} - \mu b_{k-1-j})z^{k+1-j} \\ &\qquad - 2(4 + 3\mu + 8z)z^{k-j}b_{k-2-j}\Big]c_j, \end{aligned} \tag{29}$$

where $c_j = 0$ if $j < 0$ and empty sums are zero with $c_0(z, \mu) = 1$, $c_1(z, \mu) = c_2(z, \mu) = 0$, and they satisfy the asymptotic estimate

$$c_k(z,\mu) = \mathcal{O}(\mu^{[k/3]}), \qquad |\mu| \to \infty. \tag{30}$$

Proof. The function

$$f(z,w) = \frac{e^{-Aw+Bw^2+2w/(1+\sqrt{1-2zw})}}{\sqrt{1-2zw}}\left(\frac{2}{1+\sqrt{1-2zw}}\right)^{\mu}$$

satisfies the differential equation

$$\begin{aligned} y^2(1+y)^2 f' = \big[&(2Bw - A)y^2(1+y)^2 + 2y^2(1+y) + 2zwy \\ &+ z(1+y)^2 + \mu zy(1+y)\big]f. \end{aligned}$$

Then, writing $f(z,w) = \sum_{k=0}^{\infty} c_k w^k$ we obtain the recursion (29) upon substitution. The function $f(z,w)$ can be written in the form

$$f(z,w) = e^{\mu\phi_1(z,w) + \phi_2(z,w)},$$

where ϕ_1, ϕ_2 do not depend on μ, with

$$\begin{aligned}\phi_1(z,w) &= \ln\frac{2}{1+\sqrt{1-2zw}} - \frac{1}{2}zw - \frac{3}{8}z^2w^2 \\ &= w^3\left[\frac{5}{12}z^3 + \mathscr{O}(w)\right], \qquad w \to 0\end{aligned}$$

and

$$\phi_2(z,w) = w^3\left[\tfrac{1}{6}(8z+3)z^2 + \mathscr{O}(w)\right], \qquad w \to 0.$$

Hence, (30) follows from Lemma 2.1. ∎

The first few terms are $c_0(z,\mu) = 1$, $c_1(z,\mu) = c_2(z,\mu) = 0$, and

$$\begin{aligned}c_3(z,\mu) &= \frac{z^2}{12}[(5\mu+16)z+6], \\ c_4(z,\mu) &= \frac{z^3}{64}[(35\mu+128)z+40], \\ c_5(z,\mu) &= \frac{z^4}{80}[(63\mu+256)z+70].\end{aligned}$$

PROPOSITION 5.1. *The generalized Bessel polynomials $Y_n^\mu(z)$ have the finite expansion in terms of Hermite polynomials*

$$Y_n^\mu(z) = [B(z)]^{n/2} H_n(\zeta) + n! \sum_{k=3}^{n} \frac{c_k(z,\mu)}{[B(z)]^{k/2}} \frac{H_{n-k}(\zeta)}{(n-k)!}, \qquad (31)$$

where ζ is given in (28). *This is actually an asymptotic expansion of $Y_n^\mu(z)$ for $|\mu| \to \infty$ and holds for fixed values of z and n.*

Proof. (31) follows trivially from (28) and using $c_0 = 1$, $c_1 = c_2 = 0$. The asymptotic property follows from (30) and by using $H_n(z) = \mathscr{O}(z^n)$. ∎

5.1. *Approximating the Zeros*

Let $y_{n,m}$ and $h_{n,m}$ be the mth zero of $Y_n^\mu(z)$ and $H_n(z)$, respectively, $m = 1, 2, \ldots, n$. Then, for given μ and n we can compute a first approximation of $y_{n,m}$. We obtain, inverting the relation for ζ given in (28),

$$pz^2 + qz + 1 = 0,\ p = \tfrac{1}{4}\left[\mu^2 + 4\mu + 4 + 2\zeta^2(3\mu + 8)\right],$$
$$q = \mu + 2 + 2\zeta^2.$$

This gives the relation

$$z(\zeta) = \frac{-q + i\zeta\sqrt{2(\mu + 4 - 2\zeta^2)}}{2p}.$$

Using this with $\zeta = h_{n,m}$ we obtain a first approximation of $z = y_{n,m}$.

The zeros of $Y_n^\mu(z)$ are complex, in contrast with those of the classical orthogonal polynomials, where the zeros are real and inside the domain of orthogonality. Information on the zeros distribution of $Y_n^\mu(z)$ for large values of μ seems not to be available in the literature. In Fig. 3 we show the curves $z(\zeta)$ for $\zeta \in [-\sqrt{2n+1}, \sqrt{2n+1}]$, in which interval the zeros $h_{n,m}$ of the Hermite polynomial $H_n(\zeta)$ occur [13].

6. BUCHHOLZ POLYNOMIALS

From (4) we obtain the following Cauchy-type integral for the Buchholz polynomials

$$P_n^\mu(z) = \frac{1}{2\pi i}\int_{\mathscr{C}} e^{z(\cot w - 1/w)/2}\left(\frac{\sin w}{w}\right)^\mu \frac{dw}{w^{n+1}},$$

where $\mathscr{C}$ is a circle around the origin with radius less than π.

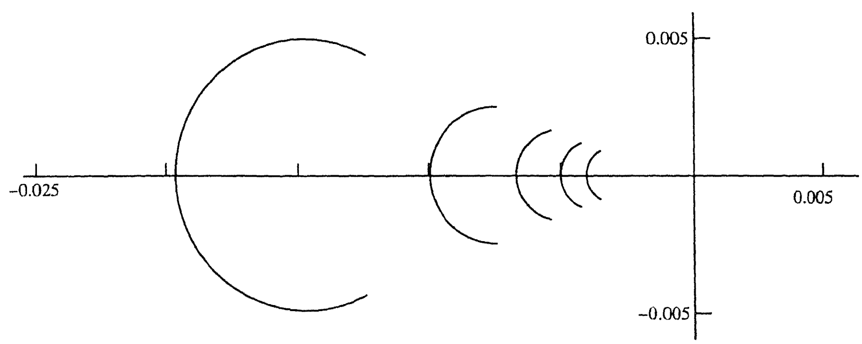


FIG. 3. The curves in the z-plane under the mapping $\zeta \to z(\zeta)$ are the images of the intervals $[-\sqrt{2n+1}, \sqrt{2n+1}]$ where the zeros of the Hermite polynomial $H_n(\zeta)$ occur. We take $n = 10$ and show the curves (from left to right) for $\mu = 100, 200, \ldots, 500$.

We have

$$P_0^\mu(z) = 1, \qquad P_1^\mu(z) = -\frac{z}{6}, \qquad P_2^\mu(z) = -\frac{1}{72}(12\mu - z^2).$$

Hence, by (12) and $p_n(z) = P_n^\mu(z)$,

$$A(z) = -\frac{z}{6}, \qquad B(z) = \frac{\mu}{6}.$$

It follows that

$$P_n^\mu(z) = \left(\frac{\mu}{6}\right)^{n/2} \sum_{k=0}^{n} \frac{c_k(z,\mu)}{(n-k)!}\left(\frac{6}{\mu}\right)^{k/2} H_{n-k}\left(\frac{-z}{2\sqrt{6\mu}}\right), \tag{32}$$

where the coefficients $c_k(z, \mu)$ of the expansion satisfy the properties given in the following lemma.

LEMMA 6.1. *The first six coefficients* $c_n(z, \mu)$ *in the expansion* (32) *are*

$$c_0(z,\mu) = 1, \qquad c_1(z,\mu) = c_2(z,\mu) = 0, \qquad c_3(z,\mu) = -\frac{z}{90},$$

$$c_4(z,\mu) = -\frac{\mu}{180}, \qquad c_5(z,\mu) = -\frac{z}{945}, \qquad c_6(z,\mu) = \frac{7z^2 - 40\mu}{113400}, \tag{33}$$

and the remaining ones satisfy, for $k \geq 1$,

$$c_k(z,\mu) = \mathscr{O}\left(\mu^{\lfloor k/4 \rfloor} + z^{\lfloor k/3 \rfloor}\right), \qquad |\mu| + |z| \to \infty. \tag{34}$$

Proof. Using equation (7), the function $f(z,w)$ of the Buchholz polynomials can be written in the form

$$f(z,w) = e^{\mu\phi_1(z,w) + \phi_2(z,w)},$$

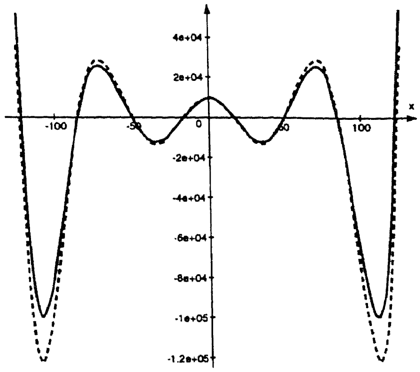
where ϕ_1, ϕ_2 do not depend on μ, with

$$\phi_1(z,w) = \ln\frac{\sin w}{w} + B(z)w^2 = w^4\left[-\frac{1}{180} + \mathscr{O}(w^2)\right], \qquad w \to 0$$

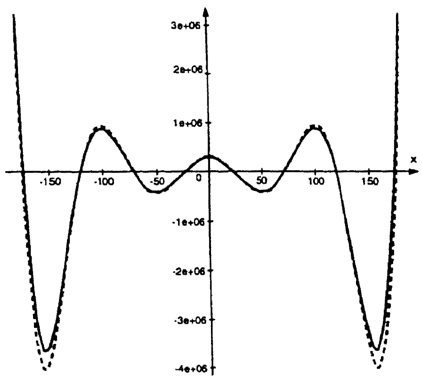
and

$$\phi_2(z,w) = z(\cot w - 1/w)/2 - A(z)w = zw^3\left[-\tfrac{1}{90} + \mathscr{O}(w^2)\right], \quad w \to 0.$$

Hence, (34) follows from Lemma 2.1. ∎



(c) $\mu = 100$



(d) $\mu = 200$

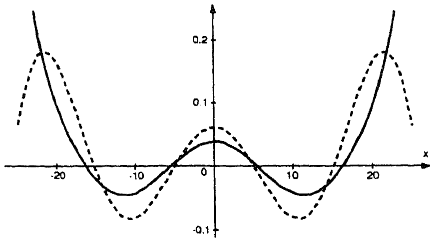
FIG. 4. Continued.

where

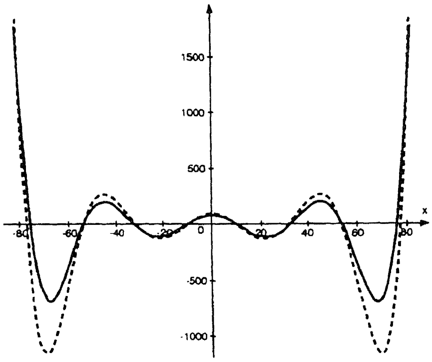
$$\zeta = -\frac{z}{2\sqrt{6\mu}}, \tag{37}$$

the first six coefficients $c_k(z, \mu)$ *are given in* (33). *This is actually an asymptotic expansion of* $P_n^{\mu}(z)$ *for* $|\mu| + |z| \to \infty$.

Proof. (36) follows trivially from (32) and using $c_0 = 1$, $c_1 = c_2 = 0$. The asymptotic property follows from Lemma 6.1 and by using $H_n(z) = \mathcal{O}(z^n)$. ∎



(a) $\mu = 10$



(b) $\mu = 40$

FIG. 4. Solid lines represent $P_{10}^{\mu}(x)$ for several values of μ, whereas dashed lines represent the right-hand side of (37) with $z = x$.

Figure 4 shows the accuracy of the approximation

$$P_n^{\mu}(z) \simeq \frac{1}{n!}\left(\frac{\mu}{6}\right)^{n/2} H_n\left(\frac{-z}{2\sqrt{6\mu}}\right) \tag{35}$$

for $n = 10$, $z = x$ and several values of μ.

PROPOSITION 6.1. *The Buchholz polynomials $P_n^{\mu}(z)$ have the finite expansion in terms of Hermite polynomials*

$$P_n^{\mu}(z) = \frac{1}{n!}\left(\frac{\mu}{6}\right)^{n/2} H_n(\zeta) + \left(\frac{\mu}{6}\right)^{n/2} \sum_{k=3}^{n} c_k(z,\mu)\left(\frac{6}{\mu}\right)^{k/2} \frac{H_{n-k}(\zeta)}{(n-k)!}, \tag{36}$$

6.1. *Approximating the Zeros*

We proceed in a similar way as in the previous cases. Let $p_{n,m}$ and $h_{n,m}$ be the mth zero of $P_n^{\mu}(z)$ and $H_n(z)$, respectively, $m = 1, 2, \ldots, n$. Then, for given μ and n we take the relation for ζ given in (37) to compute a first approximation of $p_{n,m}$ by writing

$$p_{n,m} \sim -2\sqrt{6\mu}\, h_{n,m}.$$

The accuracy of this approximation increases for increasing μ. For example, for $\mu = 20$ or 40 and $n = 10$, the relative accuracy in the zeros is $\sim 10^{-2}$. For $\mu = 100$ or 200, the relative accuracy oscillates between 10^{-2} and 10^{-3}.

7. CONCLUSIONS

Finite approximations of the generalized Bernoulli, Euler, Bessel, and Buchholz polynomials in terms of Hermite polynomials have been given. These are also asymptotic expansions of these polynomials with respect to certain sequences of the order parameter μ for $|\mu| \to \infty$. For large $|\mu|$, the nth order polynomials become, up to a factor, the nth Hermite polynomial of a certain variable. From these approximations in terms of Hermite polynomials we have obtained asymptotic estimates of the zeros of these polynomials.

ACKNOWLEDGMENTS

J. L. López wants to thank C.W.I. of Amsterdam for its scientific support and hospitality during the realization of this work. The financial support of Comisión Interministerial de Ciencia y Tecnología is acknowledged. N. M. Temme wants to acknowledge the hospitality and financial support of the Facultad de Ciencias of the Universidad de Zaragoza.

REFERENCES

1. J. Abad and J. Sesma, Computation of Coulomb wave functions at low energies, *Comp. Phys. Commun.* **71** (1992), 110–124.
2. H. Buchholz, "The Confluent Hypergeometric Function," Springer-Verlag, Berlin, 1969.
3. R. A. Buckingham, "Numerical Methods," Pitman, London, 1957.
4. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York, 1978.
5. A. Erdélyi et al., "Higher Transcendental Functions," McGraw-Hill, London, 1953.
6. E. Grosswald, "Bessel Polynomials," Lecture Notes in Mathematics **698**, Springer, New York, 1978.

7. J. L. López and J. Sesma, The Whittaker function $M_{k,\mu}$ as a function of k, *Const. Approx.* **15** (1999), 83–95.
8. J. L. López and N. M. Temme, Approximations of orthogonal polynomials in terms of Hermite polynomials, *Meth. Appl. Anal.* (in press).
9. A. T. Lundell, On the denominator of generalized Bernoulli numbers, *J. Number Theory* **26** (1987), 79–88.
10. L. M. Milne-Thomson, "The Calculus of Finite Differences (2nd ed.)," Macmillan, New York, 1965.
11. N. E. Nörlund, "Vorlesungen über Differenzenrechnung," Springer, Berlin, 1924.
12. H. M. Srivastava and P. G. Todorov, An explicit formula for the generalized Bernoulli polynomials, *J. Math. Anal. Appl.* **130** (1988), 509–513.
13. N. M. Temme, "Special Functions: An Introduction to the Classical Functions of Mathematical Physics," Wiley, New York, 1996.
14. P. G. Todorov, Une formule simple explicite des nombres de Bernoulli généralisés, *C. R. Acad. Sci. Paris Sér. I. Math.* **301** (1985), 665–666.
15. P. G. Todorov, Explicit and recurrence formulas for generalized Euler numbers, *Funct. Approx.* **22** (1993), 113–117.
16. A. Weinmann, Asymptotic expansions of generalized Bernoulli polynomials, *Proc. Camb. Phil. Soc.* **59** (1963), 73–80.
17. R. Wong and J.-M. Zhang, Asymptotic expansions of the generalized Bessel polynomials, *J. Comput. Appl. Math.* **85** (1997), 87–112.